2. The Zariski topology

2.1 Basic definitions

Big picture

→ defined affine varieties $X \subseteq \mathbb{A}^n$ as sets and polynomial functions $X \to K$

→ next: start to give more structure

⌈analogue: define $\mathbb{R}$ as set, then metric space, abelian group, field, ...⌋

→ this section: place topology on X

⇝ this determines many properties of X studied later (e.g. decomposition in components, dimension)

Refresher on basic topology: see appendix.

Def Let X be an affine variety. The Zariski topology on X is the topology whose closed sets are the affine subvarieties $Y = V_X(S) \subseteq X$, for $S \subseteq A(X)$.

Basic exercise Check that this is a topology!

Example For $X = \mathbb{A}^1$, the Zariski topology is the cofinite topol.
⇝ $Y \subseteq \mathbb{A}^1$ closed $\Leftrightarrow$ Y finite or $Y = \mathbb{A}^1$.

Fun and unnecessary exercise (Zariski topology is weird)

→ Show that every non-repeating sequence $a_1, a_2, a_3, \ldots \in \mathbb{A}^1$ (i.e. $a_n \neq a_m$ for $n \neq m$) converges to any point $a \in \mathbb{A}^1$.

→ Show that any injective map $f: \mathbb{A}^1 \to \mathbb{A}^1$ is continuous in the Zariski topology.

→ Pick a list of your favorite properties of top. spaces (Hausdorff, compact, connected, ...) and check whether $\mathbb{A}^1$ with Zariski topology has them.

→ As sets, we have $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$. Show, however, that the product topology (from Zariski-top. of $\mathbb{A}^1$) on $\mathbb{A}^1 \times \mathbb{A}^1$ is <u>not</u> the Zariski-topology on $\mathbb{A}^2$.

Hint: consider the diagonal $\Delta = \{(a,a) : a \in \mathbb{A}^1\} \subseteq \mathbb{A}^2$.
Prove that it is closed in the Zariski-topology, but not the product topology (this last claim uses something about K).

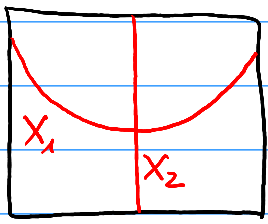
<u>Note</u> Zariski topology on $X \subseteq \mathbb{A}^n$ = relative top. of Zar. top. of $\mathbb{A}^n$
(subvarieties of X = affine varieties in $\mathbb{A}^n$ contained in X).

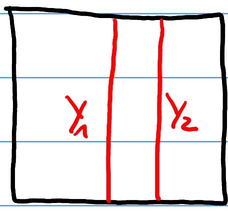
<u>Summary</u> The Zariski topology is weird, but
- → defined over every alg. closed field K
- → good enough for some purposes.

2.2. Irreducible and connected components

<u>Big picture</u>
We want to decompose an affine variety X into a union of simpler/more fundamental pieces $X = X_1 \cup X_2 \cup \cdots \cup X_r$.

<u>Example</u> Inside $\mathbb{A}^2$ consider $X = V(xy - x^3)$ and $Y = V(x^2 - 3x + 2)$





$$X = V((y - x^2) \cdot x) = \underbrace{V(y - x^2)}_{X_1} \cup \underbrace{V(x)}_{X_2}$$

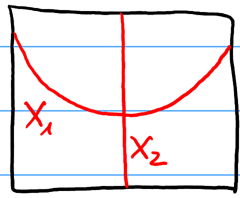
$$Y = V((x-1)(x-2)) = \underbrace{V(x-1)}_{Y_1} \cup \underbrace{V(x-2)}_{Y_2}$$

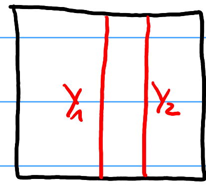
<u>Note</u>
- X looks connected, but is the union of aff. varieties X_1, X_2
- Y looks disconnected, with components Y_1, Y_2
- decompositions are visible on the algebra side as factorizations

Def (Irreducible and connected spaces)
Let X be a topological space.
(a) We say that X is reducible if it can be written as $X = X_1 \cup X_2$ for closed subsets $X_1, X_2 \subsetneq X$.
strict inclusion!
Otherwise X is called irreducible.
(b) The space X is called disconnected if it can be written as $X = X_1 \cup X_2$ for closed subsets $X_1, X_2 \subsetneq X$ with $X_1 \cap X_2 = \emptyset$.
Otherwise X is called connected.

Note disconnected $\Rightarrow$ reducible, and so irreducible $\Rightarrow$ connected.

Exa In above example:
- both X, Y are reducible
- Y is disconnected (since $Y_1 \cap Y_2 = \emptyset$).





Below we show:
- X is connected
- X_1, X_2, Y_1, Y_2 are irreducible

Note
- Intuitively, $\mathbb{A}^1$ should be irreducible, and it is for the Zariski topology (see below).
- For $K = \mathbb{C}$, taking $\mathbb{A}^1_{\mathbb{C}} = \mathbb{C}$ with the classical /Euclidean topology, we have:

$$\mathbb{C} = \{x \in \mathbb{C} : |x| \leq 1\} \cup \{x \in \mathbb{C} : |x| \geq 1\} \Rightarrow \mathbb{C} \text{ is reducible}$$

both closed, proper subsets of $\mathbb{C}$

One nice feature of irreducible spaces: open sets are big!
Exercise Let X be an irreducible space. Show that
(a) Any two non-empty open subsets $U_1, U_2 \subseteq X$ have non-empty intersection $U_1 \cap U_2$.
(b) Any non-empty open subset $U \subseteq X$ is dense (i.e. $\overline{U} = X$).
Solution [Gathmann, Remark 2.16]

Irreducible affine varieties

Big picture

- Definition of (ir)reducible & (dis)connected are easy to state, but hard to check
- Next we use tools from commutative algebra to relate these properties of X to suitable properties of its ideal $I(X)$.

Pro For an affine variety $\emptyset \neq X \subseteq \mathbb{A}^n$ the following are equivalent:

(i) X is irreducible.

(ii) $A(X)$ is a domain.

(iii) $I(X) \subseteq K[x_1, \ldots, x_n]$ is a prime ideal.

Proof Since $A(X) = K[x_1, \ldots, x_n]/I(X)$, the equivalence (ii) $\Leftrightarrow$ (iii) follows from general commutative algebra:

Lem R ring, $I \subseteq R$ ideal. Then R/I is a domain iff I is prime.

(i) $\Rightarrow$ (ii) Assume $A(X)$ is not an integral domain, and let $f_1, f_2 \in A(X)$ nonzero with $f_1 \cdot f_2 = 0$. Take $X_1 = V_X(\langle f_1 \rangle)$ and $X_2 = V_X(\langle f_2 \rangle)$.

$f_i \neq 0 \Rightarrow X_i \subsetneq X$. On the other hand:

$$X_1 \cup X_2 = V_X(\langle f_1 \rangle \cdot \langle f_2 \rangle) = V_X(\langle f_1 f_2 \rangle) = V_X(\langle 0 \rangle) = X$$

(ii) $\Rightarrow$ (i) Assume $X = X_1 \cup X_2$ is reducible with $X_1, X_2 \subsetneq X$ closed.

Since $X_i \neq X \xRightarrow[\text{Nullstellensatz}]{\text{relative}} I_X(X_i) \neq \{0\} = I_X(X)$. Choose $0 \neq f_i \in I_X(X_i) \subseteq A(X)$

Claim: $f_1 f_2 = 0$. Indeed: $f_1|_{X_1} = 0$, $f_2|_{X_2} = 0 \Rightarrow 0 = f_1 \cdot f_2|_{X_1 \cup X_2} = f_1 f_2|_X = f_1 f_2$

But $f_1 \neq 0, f_2 \neq 0$ by assumption $\Rightarrow A(X)$ is not a domain. $\square$

Cor Given an affine variety Y, there is a bijection

$$\left\{ \begin{array}{c} \text{non-empty irreducible affine} \\ \text{subvarieties in } Y \end{array} \right\} \underset{V_Y(-)}{\overset{I_Y(-)}{\rightleftarrows}} \left\{ \begin{array}{c} \text{prime ideals} \\ \text{in } A(Y) \end{array} \right\}$$

Pf Restrict the corresponding bijection $\{\text{subvar. in } Y\} \rightleftarrows \{\text{rad. ideals in } A(Y)\}$ from the relative Nullstellensatz using the proposition above. $\square$

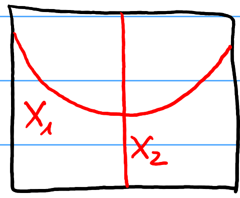
Exa

(a) A finite affine variety $Y=\{y_1,\dots,y_r\}$ is irreducible if and only if it is connected if and only if it contains at most $r\leq 1$ points.
⌈Otherwise take $Y_1=\{y_1\}$, $Y_2=\{y_2,\dots,y_r\}$ $\Rightarrow$ $Y_1\cup Y_2=Y$, $Y_1\cap Y_2=\emptyset$.⌋
In this case indeed $A(Y)=K$ is a domain.

(b) Affine space $\mathbb{A}^n$ is irreducible (and thus connected) since $A(\mathbb{A}^n)=K[x_1,\dots,x_n]$ is an integral domain.

(c) Recall the example

$$X=V(yx-x^3)=\underbrace{V(y-x^2)}_{X_1}\cup\underbrace{V(x)}_{X_2}\subseteq\mathbb{A}^2$$



We saw: X is reducible.

However: X_1, X_2 are irreducible!

- $A(X_1)=K[x,y]/\langle y-x^2\rangle\cong K[x]$ is a domain
- $A(X_2)=K[x,y]/\langle x\rangle\cong K[y]$ " " " .

TO DO LIST
☒ Show X_1, X_2, Y_1, Y_2 are irreducible
☐ Show X is connected

Similar for $Y=\underbrace{V(\langle x-1\rangle)}_{Y_1}\cup\underbrace{V(\langle x-2\rangle)}_{Y_2}$.

$\Rightarrow$ We have decomposed X into a finite union of irreducible spaces!

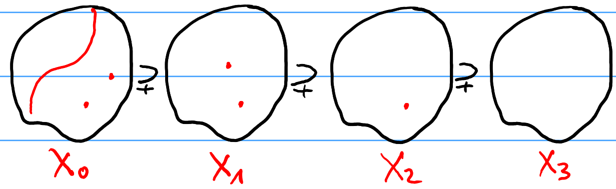
Next: study class of spaces for which such decompositions exist.

Noetherian topological spaces & irreducible decompositions

Def A topological space X is called <u>Noetherian</u> if there is no infinite strictly decreasing chain

$$X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$$

of closed subsets of X.



Exercise Show that ($\mathbb{R}^n$, Euclid. topology) is <u>not</u> Noetherian.

Lem Any affine variety X is a Noetherian topological space.

Proof Assume we had a decreasing chain $X_0 \supsetneq X_1 \supsetneq \cdots$ as above. Applying the inclusion reversing biject. $I_X(-)$ to this, we obtain an increasing chain

$$I_0 \subsetneq I_1 \subsetneq I_2 \subsetneq \cdots \subseteq A(X) \qquad (*)$$

of ideals in $A(X)$. But we have:

- $A(X) = K[x_1,\dots,x_n]/I(X)$ — contains no infinite increasing chain of ideals
- $K[x_1,\dots,x_n]$ is a <u>Noetherian ring</u>.
- Quotients of Noetherian rings are Noetherian rings.

This gives a contradiction to $(*)$. $\square$

Later, we will also consider subsets of affine varieties, e.g. open subsets such as $\mathbb{A}^n\setminus\{0\}$. These are also Noetherian:

Pro Any subset A of a Noetherian space X is also Noetherian.

Proof Assume we had a strictly decreasing sequence

$$A_0 \supsetneq A_1 \supsetneq A_2 \supsetneq \cdots \qquad (*_A)$$

of closed subsets of A (in the relative topology).

By definition: $\exists$ closed subsets $X_i \subseteq X$ with $A_i = X_i \cap A$. Then

$$X_0 \supseteq X_0\cap X_1 \supseteq X_0\cap X_1\cap X_2 \supseteq \cdots \qquad (*_X)$$

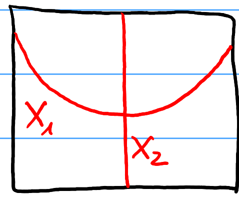
is a decreasing sequence of closed subsets of X.

To see that inclusions are strict: $X_0\cap X_1\cap\dots\cap X_{i-1}\cap \underbrace{X_i\cap A}_{=A_i \subseteq \underset{\substack{\cap\\ X_{i-1}}}{A_{i-1}} \subseteq \underset{\substack{\cap\\ X_{i-2}}}{A_{i-2}} \subseteq \cdots} = A_i$.

$\Rightarrow$ if one inclusion in $(*_X)$ was an equality, we would have $A_i = A_{i+1}$ $\lightning$ to $(*_A)$. $\square$

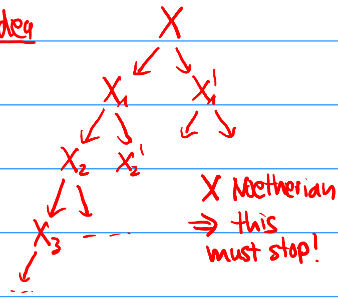
Pro (Irreducible decompositions of Noetherian spaces)
Every Noetherian topological space X can be written as a finite union $X = X_1 \cup X_2 \cup \dots \cup X_r$ of non-empty irreducible closed subsets. If one assumes $X_i \not\subseteq X_j$ for $i \neq j$, then $X_1, \dots, X_r$ are unique (up to permutation).
They are called the <u>irreducible components</u> of X.

<u>Exa</u> $X = V(yx - x^3)$
$= V(y - x^2) \cup V(x)$.



<u>Proof</u> Case $X = \emptyset$ is fine (take $r = 0 \rightsquigarrow$ empty union).
<u>Existence</u>
Assume $X \neq \emptyset$ was <u>not</u> a finite union of irreducible sets.
$\Rightarrow$ X is not irreducible (otherwise take $r=1$, $X_1 = X$).
X reducible $\rightsquigarrow$ $X = X_1 \cup X_1'$ with $X_1', X_2' \subsetneq X$ strict closed subsets
If both X_1, X_1' were finite unions of irred. sets, we would get contradict.
$\rightsquigarrow$ Wlog say X_1 is not such a union.
$\rightsquigarrow$ repeat the argument to find $X_2 \subsetneq X_1$.
$\rightsquigarrow$ By continuing: find sequence $X \supsetneq X_1 \supsetneq X_2 \supsetneq \cdots$
$\lightning$ to X Noetherian

<u>Idea</u>



<u>Uniqueness</u>
Assume we had two decompositions
$$X = X_1 \cup X_2 \cup \dots \cup X_r = X_1' \cup \dots \cup X_{r'}' \qquad (\star)$$
with $X_i \not\subseteq X_j$ and $X_i' \not\subseteq X_j'$ $\forall i \neq j$. $\qquad (\dagger)$
For $i_0 \in \{1, \dots, r\}$ fixed, we have a covering
$$X_{i_0} = X_{i_0} \cap X = (X_{i_0} \cap X_1') \cup \dots \cup (X_{i_0} \cap X_{r'}')$$
of X_{i_0} by closed sets $\xrightarrow{X_{i_0} \text{ irred.} (*)}$ $\exists j_0 : X_{i_0} = X_{i_0} \cap X_{j_0}' \Rightarrow X_{i_0} \subseteq X_{j_0}'$

(*) apply defining property several times

Repeat argument with X_{j_0}' $\Rightarrow$ $\exists i_0' : X_{j_0}' \subseteq X_{i_0'}$
But then: $X_{i_0} \subseteq X_{j_0}' \subseteq X_{i_0'} \xRightarrow{(\dagger)} i_0 = i_0'$ and so $X_{i_0} = X_{j_0}'$.
This shows: any X_i appearing on the left side of $(\star)$ also appears on the right side (and vice versa)
$\Rightarrow$ the sets $\{X_i : i = 1, \dots, r\}$ and $\{X_j' : j = 1, \dots, r'\}$ agree. $\square$

Irreducible decomposition of affine varieties

Let $X \subseteq \mathbb{A}^n$ be an affine variety with ideal $I = I(X) \subseteq K[x_1,\dots,x_n]$

Since X is Noetherian, it has an irreducible decomposition

$$X = X_1 \cup \dots \cup X_r. \qquad (*)$$

Q How to compute the decomposition $(*)$ in practice?

A Compute the primary decomposition of I:

$$I = Q_1 \cap Q_2 \cap \dots \cap Q_r \subseteq K[x_1,\dots,x_n]$$

for primary ideals Q_i. Let $P_i = \sqrt{Q_i}$, then P_i is prime.

$Q_i \subsetneq K[x_1,\dots,x_n]$

$\forall a \cdot b \in Q_i : a \in Q_i$ or $\exists m : b^m \in Q_i$

In fact:

$$I = \sqrt{I} = \sqrt{\bigcap_{i=1}^{r} Q_i} \overset{\text{Check}}{=\!=} \bigcap_{i=1}^{r} \sqrt{Q_i} = \bigcap_{i=1}^{r} P_i$$

Then

$$X = V(I) = V(P_1) \cup V(P_2) \cup \dots \cup V(P_r)$$

these are irreducible since P_i prime

Setting $X_i = V(P_i)$ and possibly removing non-minimal primes P_i
$\rightsquigarrow$ obtain irreducible decomposition $(*)$.

Another angle:

Exercise

X affine variety, then the irreducible components X_i of X are precisely the maximal irreducible subvarieties of X.

with respect to inclusion.

Hint Use the uniqueness of the irreducible decomposition.

Cor There is a bijection

$$\left\{\begin{matrix}\text{irreducible}\\ \text{components}\\ \text{of } X\end{matrix}\right\} \xrightarrow[\sim]{I_X(-)} \left\{\begin{matrix}\text{minimal}\\ \text{Prime ideals}\\ \text{of } A(X)\end{matrix}\right\}$$

Pf $\{\text{irred. subvar. of } X\}$

$\sim\downarrow I_X(-)$

$\{\text{Prime ideals of } A(X)\}$

inclusion reversing. $\square$

Connected components of affine varieties

Recall X top. space

$\Rightarrow$ X = union of its connected components

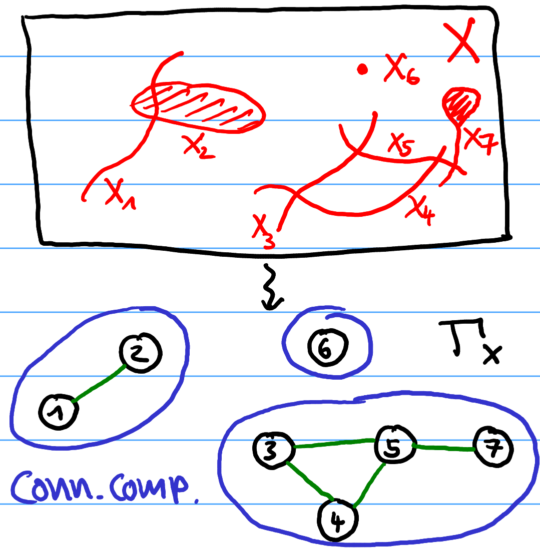
maximal connected subsets (see Appendix)

For X Noetherian, these can be obtained as follows:

Step 1 Start with the irreducible components

$$X = X_1 \cup X_2 \cup \dots \cup X_r$$

Step 2 Consider the graph Γ_X with vertices $V = \{1, \dots, r\}$ and an edge ⓘ—ⓙ if and only if $X_i \cap X_j \neq \emptyset$.



Step 3 Connected components of X

$\hat{=}$ (Path-)connected components of Γ_X

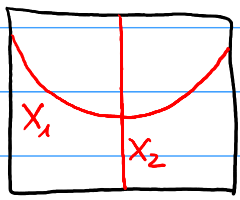
two vertices are in same component iff they are connected by a chain of edges

Exercise Prove this.

Hint See useful Lemma in Appendix on conn. components.

With this, we can finish our TO DO-List:

TO DO LIST
☒ Show X_1, X_2, Y_1, Y_2 are irreducible
☒ Show X is connected



$$X = V(yx - x^3) = X_1 \cup X_2$$

have: $X_1 \cap X_2 = V(\langle y - x^2, x \rangle)$

$$= V(\langle y, x \rangle) = \{(0,0)\} \neq \emptyset$$

$\Rightarrow \Gamma_X =$ ①—② is connected $\Rightarrow$ X connected.

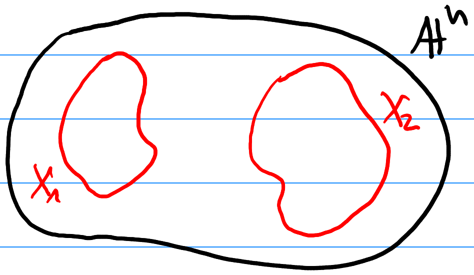
As before, we can also detect connectedness from the ring of polynomial functions $A(X)$:

Prop Let $X = X_1 \cup X_2$ be a disconnected affine variety with $X_1, X_2 \subsetneq X$ disjoint closed subsets. Then the map

$$A(X) \longrightarrow A(X_1) \times A(X_2), \quad f \mapsto (f|_{X_1}, f|_{X_2})$$

is an isomorphism.

Idea



$\rightsquigarrow$ to obtain function on X we just specify functions on X_1, X_2 <u>independently</u>.

Proof We saw before that $A(X_i) = A(X)/I_X(X_i)$, and the restriction map above is just the natural ring homomorphism

$$A(X) \longrightarrow (A(X)/I_X(X_1)) \times (A(X)/I_X(X_2)).$$

<u>Chinese Remainder Theorem</u>

R ring with ideals $I_1, I_2 \subseteq R$ that are coprime: $I_1 + I_2 = R$.
Then for $I = I_1 \cap I_2$, the map

$$R/I \longrightarrow (R/I_1) \times (R/I_2)$$

is an isomorphism.

Applying this to $R = A(X)$, $I_j = I_X(X_j)$: suffices to show $\begin{cases} I_1 \cap I_2 = \{0\} \\ I_1 + I_2 = R. \end{cases}$

- As $X = X_1 \cup X_2$, we have

$$I_X(X_1) \cap I_X(X_2) = I_X(X_1 \cup X_2) = I_X(X) = \{0\}$$

function on X vanishing on all of X is the zero function.

- As $X_1 \cap X_2 = \emptyset$, we have

$$\sqrt{I_X(X_1) + I_X(X_2)} = I_X(X_1 \cap X_2) = I_X(\emptyset) = A(X) = \langle 1 \rangle$$

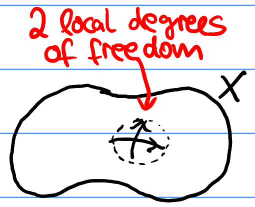
$$\Rightarrow \exists m \geq 1 : 1^m = 1 \in I_X(X_1) + I_X(X_2) \Rightarrow I_X(X_1) + I_X(X_2) = \langle 1 \rangle. \quad \square$$

Cor $A(X) = \prod_{\substack{\tilde{X} \subseteq X \\ \text{connect. comp.}}} A(\tilde{X}).$

Dimension of topological spaces

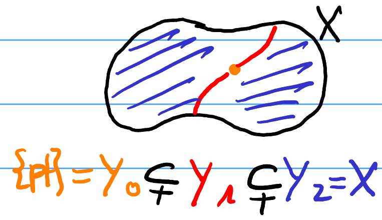
Big picture

→ For a geometric space X, an important invariant is its **dimension** $\dim(X)$



→ In many areas (linear algebra, differential geometry, ...) it is defined via the number of **(local) degrees of freedom**

→ For affine varieties, we can instead give a purely **topological characterization** via chains of irreducible subsets



Def (Dimension & codimension)

Let X be a non-empty topological space. $\mathbb{N} = \{0, 1, 2, \ldots\}$

(a) The <u>dimension</u> $\dim(X) \in \mathbb{N} \cup \{\infty\}$ is the supremum over all $n \in \mathbb{N}$ such that there is a chain

$$\emptyset \neq Y_0 \subsetneq Y_1 \subsetneq Y_2 \subsetneq \cdots \subsetneq Y_n \subseteq X$$

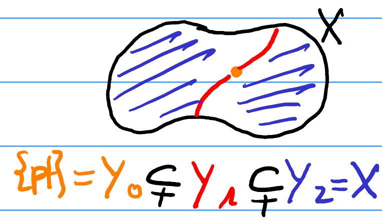
of length n of irreducible closed subsets Y_i of X.

(b) If $Y \subseteq X$ is a non-empty irreducible closed subset of X, the <u>codimension</u> $\operatorname{codim}_X Y$ of Y in X is again the supremum over all n such that there is a chain

$$Y \subseteq Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_n \subseteq X$$

of irreducible closed subsets Y_i of X containing Y

<u>Idea</u>
- Strict inclusions + irreducibility force $\dim Y_i < \dim Y_{i+1}$
- Ideally, for $\dim(X) < \infty$, a chain of maximal length in that definition should satisfy: $\dim Y_i = i$.



(In) sanity checks

(a) For $X = \mathbb{A}^1$, the closed, irreducible non-empty subsets are $\{a\}$ (for $a \in \mathbb{A}^1$) and $\mathbb{A}^1$ itself

$\Rightarrow$ $\{a\} = Y_0 \subsetneq Y_1 = \mathbb{A}^1$ is longest chain $\Rightarrow$ $\dim \mathbb{A}^1 = 1$.

(b) One might hope that any Noetherian top. space has finite dimension. This is not the case:

Exercise Let $X = \mathbb{N}$ and $\mathcal{C} = \{\emptyset, \mathbb{N}\} \cup \{\{0,1,\dots,n\} : n \in \mathbb{N}\}$.

(i) Check that $\mathcal{C}$ gives the closed sets of a topology on X.

(ii) Prove that $(X, \mathcal{C})$ is Noetherian.

(iii) Show that $\dim(X) = \infty$.

Dimension of affine varieties

When X is an affine variety, we can apply results from Commutative algebra (to its ring of functions $A(X)$) to compute $\dim(X)$.

Recall

- The Krull dimension of a ring R is the supremum of n for
$$R \supsetneq P_0 \supsetneq P_1 \supsetneq P_2 \supsetneq \dots \supsetneq P_{n-1} \supsetneq P_n$$
a chain of prime ideals P_i of R.
- The height of a prime ideal $P \subseteq R$ is the corresponding supremum of lengths n of chains
$$P = P_0 \supsetneq P_1 \supsetneq \dots \quad P_{n-1} \supsetneq P_n$$
contained in P.

Lem Let Y be a non-empty irreducible subvariety of an affine variety X.

(a) The dimension $\dim X$ equals the Krull dimension of $A(X)$.

(b) The codimension $\operatorname{codim}_Y X$ equals the height of $I_X(Y)$.

In particular: $\dim(X)$, $\operatorname{codim}_Y(X)$ are finite.

Pf Correspondence $\{\text{irr. subvar. of } X\} \rightleftarrows \{\text{prime ideals of } A(X)\}$ relates chains of subvar. to chains of prime ideals above. Their lengths are finite since $A(X)$ is a finitely generated K-algebra. $\square$

(★)

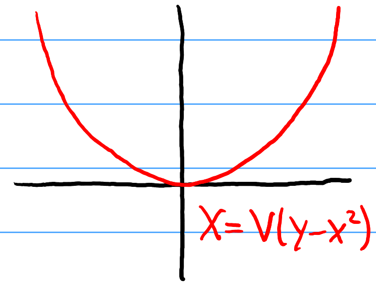
<u>Exa</u> (a) $\dim \mathbb{A}^n = n$, since the Krull dimension of $K[x_1,\dots,x_n]$ is n.

$\hookrightarrow \langle x_1,\dots,x_n\rangle \supsetneq \langle x_1,\dots,x_{n-1}\rangle \supsetneq \dots \supsetneq \langle x_1\rangle \supsetneq \{0\}$.

(b) Consider $X = V(y-x^2) \subseteq \mathbb{A}^2$

$\Rightarrow A(X) = K[x,y]/\langle y-x^2\rangle \cong K[x]$

$\overset{(★)}{\Rightarrow} \dim(X) = 1$.



Next: transfer results about dimension from commutative algebra.

<u>Pro</u> (Properties of dimension)

Let X, Y be non-empty irreducible affine varieties.

(a) We have $\dim(X\times Y) = \dim X + \dim Y$, with $X\times Y$ having the Zariski topology. ↜ another way to see $\dim \mathbb{A}^n = n$

(b) If $Y \subseteq X$ we have $\dim X = \dim Y + \operatorname{codim}_X Y$.

In particular, $\operatorname{codim}_X\{a\} = \dim X$ for every point $a \in X$.

(c) If $f \in A(X)$ is non-zero, every irreducible component of $V_X(f)$ has codimension 1 in X (and thus dimension $\dim X - 1$).

<u>Proof</u>

(a) Have seen: $A(X\times Y) = A(X) \otimes_K A(Y)$

So we need to show: $\dim A(X)\otimes_K A(Y) = \dim A(X) + \dim A(Y)$.

<u>Lem</u> R, S fin. gen. algebras over K and domains

$\Rightarrow \dim R\otimes_K S = \dim R + \dim S$.

<u>Pf. idea</u> For $n = \dim R$, $m = \dim S$ choose Noether normalizations

$K[x_1,\dots,x_n] \hookrightarrow R$ and $K[y_1,\dots,y_m] \hookrightarrow S$ ↘ target = f.g. module/domain

$\overset{\text{check}}{\Longrightarrow}$ $K[x_1,\dots,x_n,y_1,\dots,y_m] \hookrightarrow R\otimes_K S$ also Noether norm. □

(b) uses that all maximal chains of prime ideals in $A(X)$ have the same length.

$$\underbrace{\underbrace{P_n \supsetneq \dots \supsetneq P_y}_{\text{length } \dim Y} \supsetneq \underbrace{P_{m-1} \supsetneq \dots \supsetneq P_0}_{\text{length } \operatorname{codim}_Y X}}_{\text{length } \dim X}$$

(c) is Krull's principal ideal theorem. □

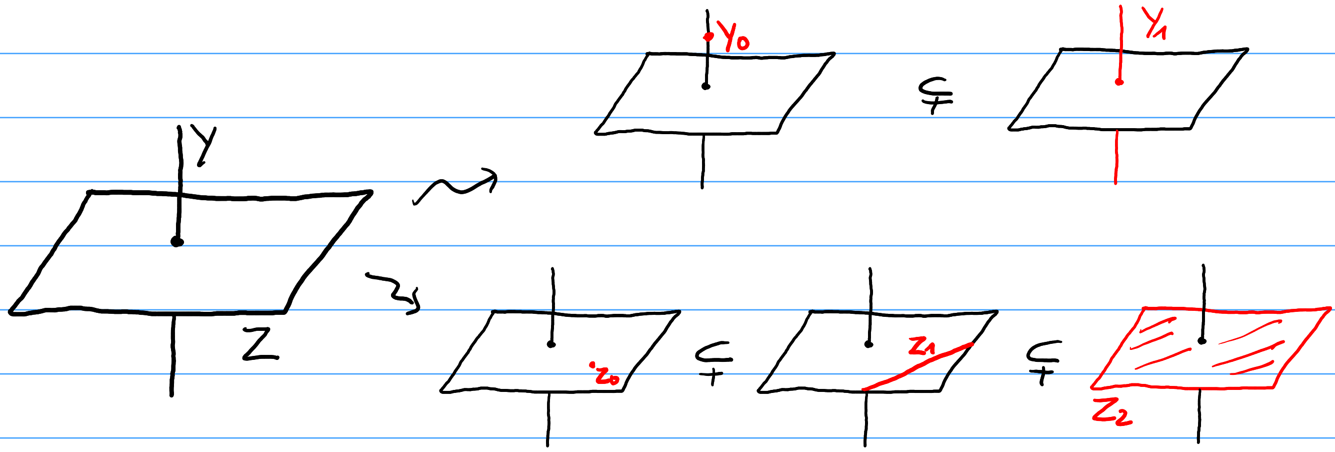
(comp. of $V(f)$ $\hat{=}$ min. primes in $A(X)/\langle f\rangle$).

Dimension theory of reducible spaces

Many results above formulated for X irreducible aff. var.

<u>Question</u> What about reducible X?

<u>Exa</u> $X = V(x_1x_3, x_2x_3) = \underbrace{V(x_1, x_2)}_{=Y} \cup \underbrace{V(x_3)}_{=Z} \subsetneq \mathbb{A}^3 \rightsquigarrow$ $\dim Y = 1$, $\dim Z = 2$



Pictures show two maximal ($\rightarrow$ cannot be extended further) chains of irred. subsets

$\Rightarrow \operatorname{codim}_X Y_0 = 1$ and $\operatorname{codim}_X Z_0 = 2$. $\rightsquigarrow$ local dimension of X at Y_0, Z_0

Guess: $\dim X = 2$.

To make this precise:

<u>Exercise</u> Let X be a top. space and $A \subseteq X$ any subset. Show: $\dim A \leq \dim X$.

<u>Exercise</u> Let X be a Noetherian space and $Y \subseteq X$ irreducible. Show that there exists an irred. comp. of X containing Y.

<u>Pro</u> (a) If $X = X_1 \cup \dots \cup X_r$ is the irreducible decomposition of a Noetherian topological space, then
$$\dim X = \max\{\dim X_1, \dots, \dim X_r\}$$
(b) We always have
$$\dim X = \sup\{\operatorname{codim}_X\{a\} : a \in X\}.$$

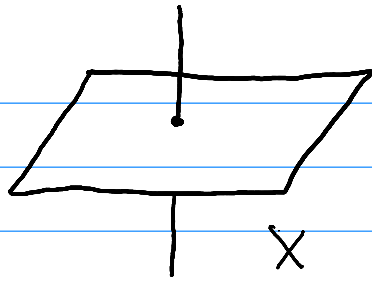
<u>Pf</u> (a) "$\geq$" Since $X_i \subseteq X \xRightarrow{\text{Exercise}} \dim X_i \leq \dim X$

"$\leq$" If $\dim X \overset{(*)}{\geq} n$ we find a chain $Y_0 \subsetneq \dots \subsetneq Y_n \subseteq X$ of irred. subsets

$\overset{\text{Exercise}}{\Rightarrow} \exists i : Y_n \subseteq X_i \Rightarrow \dim X_i \geq n \Rightarrow \dim X = \max\{n : (*)\} \leq \max\{\dim X_i : i\}$.

(b) Exercise (or see [Gathmann, Remark 2.31]. $\square$

While spaces like X above are nice for illustration, it is often possible to prove stronger results for spaces whose irred. components all have the same dimension.



Def (Pure-dimensional spaces)

(a) A Noetherian top. space X is said to be of pure dimension n if every irreducible component of X has dimension n.

(b) An affine variety X is called
- a curve if it is of pure dimension 1.
- a surface if it is of pure dimension 2.
- a hypersurface in a pure-dimensional affine variety Y if $X \subseteq Y$ is an affine subvariety of pure dimension $\dim Y - 1$.

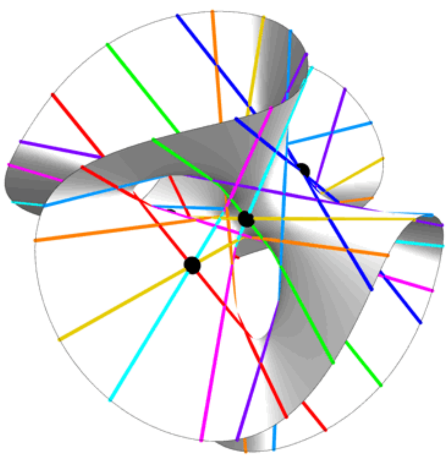
Fancy example Given parameters $a, b, c, \dots \in K$, consider the cubic surface

$$S = V(a \cdot x_1^3 + b \cdot x_1^2 x_2 + c \cdot x_1^2 x_3 + \dots + p x_3^2 + q \cdot x_1 + r \cdot x_2 + s \cdot x_3 + t) \subseteq \mathbb{A}^3$$

Unless $a = \dots = s = 0 \rightsquigarrow S$ is a non-empty hypersurface and a surface

Cool fact For most* values of $a, b, \dots,$ there are exactly $\boxed{27 \text{ lines}}$ $L_1, L_2, \dots, L_{27} \subseteq \mathbb{A}^3$ contained in S

The union $L = L_1 \cup \dots \cup L_{27}$ is a reducible curve



27 Lines on a Cubic Surface – Greg Egan

* the set of $(a, b, \dots, s, t) \in \mathbb{A}^{20}$ with the above property is non-empty and Zariski open.

$\Leftrightarrow$ the set where the property fails is Zariski closed and not all of $\mathbb{A}^{20}$, thus cut out by non-trivial system of equations in $a, b, \dots, t$.

Hypersurfaces and unique factorization domains

Have seen: X irred. affine var., $Y = V_X(f)$, $0 \neq f \in A(X)$
$\Rightarrow Y \subseteq X$ is hypersurface $\quad (\dim Y = \dim X - 1)$

Question What about "$\Leftarrow$"?

Answer depends on whether $A(X)$ is a unique factorization domain!

Recall An integral domain R is a unique factorization domain (UFD) if any $0 \neq x \in R$ can be written as
$x = u \cdot p_1 \cdot \ldots \cdot p_n$ with $n \geq 0$, for u unit, p_i irreducible,
and the p_i are unique up to permutation & mult. by units.

Pro Let R be a Noetherian integral domain (e.g. $R = A(X)$, X irred. aff. var.).
Then the following are equivalent:
(a) Every prime ideal of height / codimension 1 in R is principal.
(b) R is a unique factorization domain.

Note For $R = A(X)$: prime ideals $P \subseteq A(X)$ of height 1 $\widehat{=}$ irreducible hypersurfaces $Y = V_X(P) \subseteq X$ $\Big\} \Rightarrow$ (a) says: $Y = V_X(f)$

Proof "$\Rightarrow$" $0 \neq f \in R \Rightarrow$ either f irred. or $f = f_1 \cdot f_2$
Continue inductively decomposing f_1, f_2 into irred. factors
Since R Noetherian $\xrightarrow[\text{as irred. decomp.}]{\text{similar argument}}$ $f = f_1 \cdot f_2 \cdot \ldots \cdot f_n$ for f_i irreducible

$\boxed{\text{Claim } f_i \text{ are prime}}$; This then implies uniqueness part of UFD-def.:
If we have: $f = f_1' \cdot f_2' \cdot \ldots \cdot f_{n'}'$ a second factoriz. $\Rightarrow \forall i: f_i \mid f_j' \mid f_k \Rightarrow f_i \sim f_j'$ up to units.

proof

Let $P \subseteq R$ minimal prime containing f_i $\xrightarrow[\text{PID theorem}]{\text{Krull's}}$ $\operatorname{codim} P = 1$
$\xRightarrow{(a)}$ $P = \langle g \rangle$ principal, and so g prime
$f_i \in P \Rightarrow g \mid f_i$, but f_i irred. $\Rightarrow f_i = u \cdot g$, u unit $\Rightarrow f$ prime.

"$\Leftarrow$" $P \subseteq R$ prime codim 1 $\Rightarrow \{0\} \subsetneq P \subsetneq R$ so choose $0 \neq f \in P$
if $P = \langle 0 \rangle$ then proof finished already.
As $P \neq R \Rightarrow f$ is not a unit
(b): R is UFD $\Rightarrow f = f_1 \cdot \ldots \cdot f_k$, $f_i \in R$ irred. (in fact: prime)
P is prime, $f \in P \Rightarrow \exists i: f_i \in P \Rightarrow \underbrace{\{0\}}_{\text{prime}} \subsetneq \underbrace{\langle f_i \rangle}_{\text{prime}} \subseteq P$
$\operatorname{codim} P = 1 \Rightarrow \langle f_i \rangle = P$
$\Rightarrow P$ principal. $\square$

<u>Example/Exercise</u> Let $R = K[x_1,x_2,x_3,x_4]/\langle x_1x_4 - x_2x_3\rangle$. Show:

(a) R is an integral domain of dimension 3.

(b) $x_1,\dots,x_4$ are irreducible, but not prime in R. ($\Rightarrow R$ is not UFD)

(c) x_1x_4 and x_2x_3 are decompos. of the same elem. in R that do not agree up to permutation and units.

(d) $\langle x_1,x_2\rangle$ is a prime ideal of codim 1 in R that is not principal.

$\Rightarrow Y = V(x_1,x_2) \subseteq V(x_1x_4 - x_2x_3) = X$ is hypersurface in X whose ideal cannot be generated by one elem. in $A(X)$.

<u>Hint</u> Use the grading of $K[x_1,x_2,x_3,x_4]$ (or: Zariski tangent spaces)

<u>In fact</u> Y is also not cut out as a set by a single elem. in $A(X)$ (there is no $g \in A(X)$: $Y = V_X(g)$)

see: [Vakil, The rising sea, Exercise 15.4.T].

<u>Application</u> Hypersurfaces in $\mathbb{A}^n$.

$X \subseteq \mathbb{A}^n$ hypersurface, $X = X_1 \cup \dots \cup X_r$ irred. decomposition.

Since $R = A(\mathbb{A}^n) = K[x_1,\dots,x_n]$ is a UFD

$\xRightarrow{\text{Pro}} I(X_i)$ is codim. 1 prime ideal, so $I(X_i) = \langle f_i\rangle$, $f_i \in R$ prime

$\Rightarrow I(X) = I(X_1 \cup \dots \cup X_r) = \underbrace{I(X_1)}_{\langle f_1\rangle} \cap \dots \cap \underbrace{I(X_r)}_{\langle f_r\rangle} = \langle \underbrace{f_1 \cdot f_2 \cdot \dots \cdot f_r}_{=:f}\rangle$

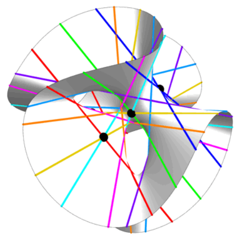
$\Rightarrow I(X) = \langle f\rangle$ principal and f is unique / units.

<u>Def</u> (Degree of affine hypersurface)

Let $X \subseteq \mathbb{A}^n$ be a hypersurface and $I(X) = \langle f\rangle$, $f \in K[x_1,\dots,x_n]$.

Then the <u>degree</u> of X (denoted $\deg X$) is defined as $\deg X = \deg f$.

$\deg X$	
1	linear hypersurface (or <u>hyperplane</u>)
2	quadratic "
3	cubic "
$\vdots$	$\vdots$

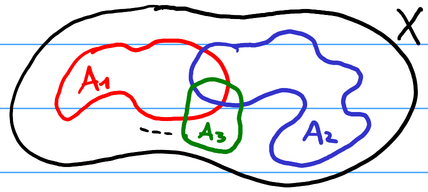


Appendix: Connected components

Def Let X be a topological space. A connected component of X is a maximal connected subset of X.

Pro X is the disjoint union of its connected components.

Crucial exercise X top. space,



$A_i \subseteq X$ connected $(i \in I)$ with $\bigcap_{i \in I} A_i \neq \emptyset$

$\Rightarrow \bigcup_{i \in I} A_i$ connected.

Proof of Proposition

For $x \in X$ one has that $\bigcup_{\substack{A \subseteq X \\ \text{connected}, x \in A}} A =: \widetilde{X}_x$ is a connected comp., $x \in \widetilde{X}$

If $\widetilde{X}, \widetilde{X}'$ are conn. comp. with $\widetilde{X} \cap \widetilde{X}' \neq \emptyset \Rightarrow \widetilde{X} \cup \widetilde{X}'$ connect.

By maximality: $\widetilde{X} = \widetilde{X} \cup \widetilde{X}' = \widetilde{X}'$. $\square$

Appendix: Basics of topology

$\{A : A \subseteq X\}$ power set

Def A <u>topology</u> on a set X is the data of a set $\mathcal{C} \subseteq \mathcal{P}(X)$ of subsets of X (called the <u>closed sets</u> of the topology) such that

(a) $\emptyset$ and X are closed

(b) arbitrary intersections $\bigcap_{i \in I} Y_i$ of closed sets Y_i are closed

(c) finite unions $Y_1 \cup \dots \cup Y_r$ of closed sets are closed.

A set $U \subseteq X$ is called <u>open</u> if $X \setminus U$ is closed.

The pair $(X, \mathcal{C})$ is then called a <u>topological space</u>

Note Often a topology is specified by its set of opens $\mathcal{U} \subseteq \mathcal{P}(X)$.

In algebraic geometry: specifying $\mathcal{C}$ is more natural

$\rightsquigarrow$ standard axioms for $\mathcal{U}$ are obtained from (a)–(c) above by taking complements in X.

Examples Let X be a set.

(a) Trivial topology: $\mathcal{C} = \{\emptyset, X\}$

(b) Discrete topology: $\mathcal{C} = \{A : A \subseteq X\} = \mathcal{P}(X)$

(c) Cofinite topology: $\mathcal{C} = \{A : A \subseteq X \text{ finite}\} \cup \{X\}$

(d) Euclidean topology on $X = \mathbb{R}^n$: $\mathcal{U} = \{U \subseteq \mathbb{R}^n : \forall x \in U\ \exists \delta > 0 : B_\delta(x) \subseteq U\}$

Constructions X topological space, $A \subseteq X$ subset

→ The <u>closure</u>

$$\bar{A} = \bigcap_{\substack{Y \subseteq X \text{ closed} \\ A \subseteq Y}} Y .$$

is the smallest closed subset of X containing A.

→ The <u>subspace topology</u> on A is defined by

$\mathcal{C}_A = \{B \subseteq A : \exists Y \subseteq X \text{ closed with } B = A \cap Y\}$.

If A closed itself: $B \subseteq A$ closed $\iff$ $B \subseteq X$ closed & contained in A.

X, Y topological spaces

→ $f: X \longrightarrow Y$ is continuous if for all $Z \subseteq Y$ closed, the preimage $f^{-1}(Z) \subseteq X$ is closed.

→ $X \times Y$ has the product topology : $U \subseteq X \times Y$ open if it is a finite union of sets $U_i \times V_i$ for $U_i \subseteq X$, $V_i \subseteq Y$ open.